Hybrid Systems Modeling for Robots with Contact

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What do we mean by “Hybrid System”

\[ H := (\mathcal{J}, \Gamma, D, F, G, R) \]

Motivation: Why hybrid systems modeling?

- Insight into design, control
- Compare different systems
- Understand the problem structure
- Analysis of the dynamics
  - Provable properties
  - Parametric dependence
- Differential analysis
  - Identifying non-smooth points

\[
\frac{\partial ((U\lambda)_{2n} - (U\lambda)_{1n})}{\partial \theta_d} \bigg|_{\theta_d=0} = \frac{2 \rho \tau \sin^2(\theta_m)}{\ell^2 \cos(\theta_m)} < 0
\]


Robots with Changing Contact Conditions


When should you not look at contacts?

J. Bender and A. Schmitt, “Constraint-based collision and contact handling using impulses,” CASA, 2006
What do we mean by “Hybrid System”

\[ \mathcal{H} := (\mathcal{J}, \Gamma, \mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R}) \]


Standard Definition of Hybrid System

How Standards Proliferate:
(See: A/C chargers, character encodings, instant messaging, etc.)

Situation:
There are 14 competing standards.

14?! Ridiculous!
We need to develop one universal standard
that covers everyone's use cases.
Yeah!

Situation:
There are 15 competing standards.

https://xkcd.com/927/
The (Self-)Manipulation Hybrid System

$\mathcal{H} := (\mathcal{I}, \Gamma, \mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R})$

Specifically, we want a model that has:

- Simple physical assumption (e.g. plastic impact, massless legs)
- Disjoint domains of varying dimensions
- Guards with arbitrary co-dimension (not just on boundary)
- Certain analytic and geometric structure (e.g. hybrid, corners)
- Consistency properties
  - Deterministic (and that the guards are disjoint)
  - Non-blocking
  - Finite ($\leq 2$) transitions at a time $t$
  - Unique execution from any mode (i.e. can’t get stuck in the wrong mode)

Physical Assumptions

- Simplifying physical assumptions:
  - Rigid bodies
  - Plastic impact
  - Persistent contact
  - Coulomb friction
  - Massless legs

All models are wrong, but some models are useful

Massless Dynamics

• Typical impulse calculation inverts the inertia tensor:
\[ \hat{P}_J = (A_J \overline{M}^{-1} A_J^T)^{-1} A_J \hat{q} \]

• Instead solve the dynamics and constraints simultaneously:
\[
\begin{bmatrix}
\dot{q} \\
\lambda
\end{bmatrix} = \begin{bmatrix}
\overline{M} & A^T \\
A & 0
\end{bmatrix}^{-1} \begin{bmatrix}
\gamma - \overline{N} \\
0
\end{bmatrix} - \begin{bmatrix}
\overline{M} & A^T \\
A & 0
\end{bmatrix}^{-1} \begin{bmatrix}
\overline{C} \\
\dot{\Lambda}
\end{bmatrix} \dot{q}
\]

• Now the constrained impulse is well defined:
\[ \hat{P}_J := -\Lambda_J A_J \hat{q} \]
\[
\begin{bmatrix}
\overline{M}_J^+ & A_J^{+T} \\
A_J^+ & \Lambda_J
\end{bmatrix} := \begin{bmatrix}
\overline{M} & A_J^T \\
A_J & 0_{J \times J}
\end{bmatrix}^{-1}
\]

Complementarity Constraints

- **Impulse/Velocity Comp.:**
  \[
  U_j(\hat{P}_J) \geq 0, \quad A_j q^+ = 0 \quad \forall j \in J
  \]
  \[
  U_k(\hat{P}_J) = 0, \quad A_k q^+ > 0 \quad \forall k \in \mathcal{I} \setminus J
  \]

- **Alternate Formulation**
  \[
  U_j(\hat{P}_J) \geq 0, \quad A_j q^+ = 0 \quad \forall j \in J
  \]
  \[
  U_k(\hat{P}_J) = 0, \quad U_k(\hat{P}_{J \cup \{k\}}) < 0 \quad \forall k \in \mathcal{I} \setminus J
  \]

- **Or, in a unified form:**
  \[
  (k \in J) \iff (U_k(\hat{P}_{J \cup \{k\}}) \geq 0) \quad \forall k \in \mathcal{I}
  \]

- **Same for Force/Acceleration:**
  \[
  U_j(\lambda_J) \geq 0, \quad A_j \ddot{q} + A_j \dot{q} = 0 \quad \forall j \in J
  \]
  \[
  U_k(\lambda_J) = 0, \quad U_k(\lambda_{J \cup \{k\}}) < 0 \quad \forall k \in \mathcal{I} \setminus J
  \]
  \[
  (k \in J) \iff (U_k(\lambda_{J \cup \{k\}}) \geq 0) \quad \forall k \in \mathcal{I}
  \]

Spurious Transitions

- Even once these modeling assumptions are rectified, problems still arise
- Impacts at one contact point can cause others to lift off
- Similar to the rocking block Zeno problem
- The pseudo-impulse couples in continuous forces to truncate these transitions

A point mass impacting a hill will transition to sliding up it
• For any steep angle $\theta < 90$
• For any arbitrarily slow speed
• A free body diagram of the impact makes this clear
• Add a pseudo-impulse proportional to the continuous time forces
• The pseudo-impulse allows the mass to come to a rest

The (Self-) Manipulation Hybrid System

\[ \mathcal{H} := (\mathcal{I}, \Gamma, \mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R}) \]

\[ \mathcal{I} := \{I, J, \ldots, K\} \subset \mathbb{N} \]

\[ \Gamma \subset \mathcal{I} \times \mathcal{I} \]

\[ \mathcal{D} := \bigsqcup_{I \in \mathcal{I}} D_I \]

\[ \mathcal{F} : \mathcal{D} \to T\mathcal{D} \]

\[ F_I := \mathcal{F}|_{D_I} \]

\[ \mathcal{G} := \bigsqcup_{(I,J) \in \Gamma} G_{I,J} \]

\[ G_{I,J} \subset D_I \]

\[ \mathcal{R} : \mathcal{G} \to \mathcal{D} \]

\[ R_{I,J} := \mathcal{R}|_{G_{I,J}} : G_{I,J} \to D_J \]

- Key idea: Build the hybrid system out of “hybrid” parts:
  - Domain and guards are hybrid manifolds (with corners), i.e. a finite disjoint union of manifolds w/corners
  
  \[ \bigsqcup_{I \in \mathcal{I}} M_I = \bigcup_{I \in \mathcal{I}} \{I\} \times M_I = \{(J, x) : J \in \mathcal{I}, x \in M_J\} \]

- The flow is a hybrid vector field on the hybrid tangent bundle

- An execution runs on a hybrid time domain,
  
  \[ \chi : \mathcal{T} \to \mathcal{D} \]

  \[ \mathcal{T} = \bigsqcup_{i=1}^{N} T_i \]

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Why hybrid components?

\[ T = \{ (-\infty, t], [t], [t, \infty) \} \]
The (Self-)Manipulation Hybrid System

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- Specifically, we want a model that has:
  - Simple physical assumption (e.g. plastic impact, massless legs)
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Conclusions

- Better simple models
- Specific physical assumptions
  - Massless legs
  - Pseudo-impulse (more plastic than plastic)

- Hybrid system model structure

\[ \mathcal{H} := (\mathcal{J}, \Gamma, \mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R}) \]
\[ \mathcal{J} := \{I, J, \ldots, K\} \subset \mathbb{N} \]
\[ \Gamma \subset \mathcal{J} \times \mathcal{J} \]
\[ \mathcal{D} := \bigsqcup_{I \in \mathcal{J}} D_I \]

\[ \mathcal{F} : \mathcal{D} \rightarrow TD \]
\[ F_I := \mathcal{F}|_{D_I} \]

\[ \mathcal{G} := \bigsqcup_{(I, J) \in \Gamma} G_{I, J} \]
\[ G_{I, J} \subset D_I \]

\[ \mathcal{R} : \mathcal{G} \rightarrow \mathcal{D} \]
\[ R_{I, J} := \mathcal{R}|_{G_{I, J}} : G_{I, J} \rightarrow D_J \]

Convergence/Divergence of Contacts

Relating different models

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**Modeling hierarchy** (Mathematical or physical)

- **TEMPLATES**
  - Simple
  - General
  - Prescriptive
  - Control guide

- **ANCHORS**
  - Elaborate
  - Representative (less overconstrained)

- **ORGANISM**
  - Complex
    - (many degrees of freedom, high dimensionality)
  - Redundant
    - Joints
    - Muscles
    - Neurons

**Nature of system**

**Legged land locomotion**

- Spring-loaded inverted pendulum (SLIP)
- Lateral leg spring (LLS)
- Multiple legs, joints and muscles
- Multiple legs, joints and muscles

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More complicated contact
Thank You

P.S. Ask me about PhD and faculty positions at CMU!
Thank you to the ARL/GDRS RCTA project.